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Journal of Sound and Vibration 275 (2004) 1–16
JOURNAL OF SOUND AND VIBRATION
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A method to determine the periodic solution of the non-linear dynamics system

Dexin Li, Jianxue Xu*

The State Key Laboratory of Mechanical Strength and Vibration, School of Architectural Engineering and Mechanics, Xi'an Jiaotong University, Xi'an 710049, PR China

Received 11 December 2002; accepted 19 June 2003

Abstract

In this paper, the generalized shooting method and the harmonic balancing method to determine the periodic orbit, its period and the approximate analytic expression of the non-linear bearing–rotor system are presented. At first, by changing the time scale, the period of the periodic orbit of the non-linear system is drawn into the governing equation of the system explicitly. Then, the generalized shooting procedure is sought out. The increment value changed in the iteration procedure is selected by using the optimization method. The procedure involves determining the periodic orbit and its period of the system, and the stability of the periodic solution is determined by using Floquet stability theory. The validity of such method is verified by determining the periodic orbit and period of the forced van der Pol equation. Secondly, the periodic solution of the non-linear rotor–bearing system is expanded into Fourier series according to the character of the solution obtained by using the generalized shooting method. Then the approximate analytic expression of the periodic solution of the system is obtained by using the harmonic balancing method. Theoretically, the solution with any precision can be obtained by adding the number of the harmonics. At last, the periodic orbit, period and approximate analytic expressions of the periodic solution of the non-linear rotor–bearing system are provided.

1. Introduction

The method of determining the periods of the periodic orbits of the non-linear dynamics systems is one of the most important fields in the non-linear research, because it is related to many important problems such as bifurcation, stability problem, chaos and so on. There are many ways to determine the periodic orbit and period of the non-linear dynamics system. Those methods

*Corresponding author. Tel.: +86-29-2312336; fax: +86-29-2312336.
E-mail address: huaqun@pub.xaonline.com (J. Xu).

0022-460X/$ - see front matter
doi:10.1016/S0022-460X(03)00656-4

generally can be classified into two categories, namely, the frequency domain methods and the time domain methods. The former includes the harmonic balancing methods, and the latter includes the shooting method and Poincare method. In 1965, Urabe [1] used the oscillation method for the non-linear periodic system. In 1966, Urabe and Beiter [2] extended his investigation, and presented the numerical computation of the non-linear forced oscillations. In 1983, Lau et al. [3] gave an incremental harmonic balance method with multiple time scales for a periodic vibration of the non-linear system. In 1994, Xu and Cheng [4] investigated the harmonic balancing method, and presented a new approach to solve a type of the vibration problem. In 1997, Sundararajan and Noah [5] presented a shooting algorithm with the pseudo-arc length continuation for a non-linear dynamic system subjected to the periodic excitation. The frequency-domain methods assume a periodic solution with a finite number of the harmonics and then estimate the coefficients of the assumed harmonics by substituting the periodic solution in the equations of motion given and solve the resulting set of the non-linear algebraic equations, which may be polynomial, exponential or transcendental. The number of nonlinear equations increases two with each addition of harmonic terms in assumed solutions and the total number of equations to be solved enormously exceeds that of the order of the original system. The methods become tedious even for moderately large order systems and the convergence of such a huge set of the non-linear algebraic equations poses problems since it is largely dependent on the initial guesses. A time domain method, such as the shooting method, assumes a point in the periodic solution and then shoots in time for an assumed period of T and checks if the periodicity condition is satisfied. The precision required for convergence is calculated by solving a linear set of algebraic equations. The non-linear algebraic equations that arise from the shooting procedure are of the same order as that of the original system and, hence, are considerably easier to solve than the larger system of polynomial/exponential/transcendental equations that result from the harmonic balance methods. Fortunately the higher the order of the system, the more evident the advantage of the shooting method. The classical perturbation method is only suitable to the weak non-linear system [6]. For the continuous method, in order to pass through a turning point that generally involves a stable and an unstable branch, a shooting algorithm with pseudo-arc length continuation was presented for the non-linear dynamic system subjected to periodic excitation in paper [5]. Unfortunately it was based on the known period of an excitation. As to a non-autonomous system, the period of the responses can be determined through the methods mentioned above, because it is related to the period of the excitation. If there is a period that has nothing to do with the excitation in the non-linear dynamics system, there will be some difficulties when using these methods. If there is a periodic orbit in an autonomous system, the question is how to determine it quickly. Although it can be obtained through Runge–Kutta numerical integration method, there is blindness to some extent, and the error of the period solved cannot be less than the integration step-size.

In order to overcome the shortcomings mentioned above, the traditional shooting method is modified using the period of the system as a parameter in this paper. As the result, the periodic orbit and period of the non-linear dynamics system is determined efficiently. The efficiency in determining the periodic orbit and its period of the non-linear bearing–rotor system is demonstrated in this paper. On the basis of determining the period (or frequency) of the system, the periodic solution of the non-linear bearing–rotor system is expended into Fourier series concerned with frequency solved above. Then the approximate analytic expressions of the periodic solution of the system are obtained by using the harmonic balancing method. In theory, the solution with any precision can be obtained by adding the number of the harmonics. Finally, the periodic orbit, period and the approximate analytic expressions of the solution of the non-linear rotor–bearing system are obtained.

2. Method analysis

2.1. The generalized shooting method

Consider the following non-linear system:

$$\frac{dx}{dt} = f(x, t, \alpha), \quad x, f \in R^n, \ t, \alpha \in R, \tag{1}$$

where α is a physical parameter. Suppose that a periodic orbit of system (1) exists and its period is T. The periodic orbit is

$$x^p = x^p(t + T), \quad x \in R^n, \ t \in R. \tag{2}$$

It can be seen from Eq. (1) that the period T is not explicitly shown in the governing equation of the system. In order to show the period T explicitly in the equation, system (1) is transformed into the following equation by using $t = T\tau$:

$$\frac{dx}{d\tau} = Tf(x, T\tau, \alpha), \quad x, f \in R^n, \ \tau, \alpha \in R, \tag{3}$$

where α is a physical parameter and T is the original period of system (1). The period of the periodic orbit of system (3) is changed to 1, namely $x(\tau) = x(\tau + 1)$. The period of system (1) is an unknown value. However, the period of system (3) is a known value. Integrate system (3) from $\tau = 0$ to 1, which corresponds to integrating system (1) from $t = 0$ to T. Because the period T is already explicitly present in the equation of system (3), the period as a parameter take part in the iteration procedure of the shooting method. The periodic orbit and its period of system (1) can be obtained indirectly by determining that of system (3). The problem of determining the periodic orbit and period of system (1) is transformed to that of finding those of system (3). For the periodic solution of system (3), we have $x^0 = x^1$, where x^0 denotes the state vector at the time 0, and x^1 denotes the state vector at the time 1 of system (3).

Choose the initial condition as follows:

$$\begin{cases} x_i^0 = \eta_i, \\ T^0, \end{cases} \quad i = 1, 2, \ldots, n. \tag{4}$$

Choose the initial condition (4) and integrate system (3) from $\tau = 0$ to 1. Then the value of x^1 can be obtained. Apparently, the value of x^1 is dependent on the initial condition chosen, x^1 is a function of η and T^0. Thus we can speak of $x^1(\eta, T^0)$, with $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$.

r is the value of x^1 obtained after integrating one period minus x^1 chosen as the initial value. Certainly, r is also a function of η and T^0:

$$r_i = x_i^0 - x_i^1, \quad i = 1, 2, \ldots, n. \tag{5}$$

For the periodic solution of the system, the following condition must be satisfied:

$$r_i = x_i^0 - x_i^1 = 0, \quad i = 1, 2, \ldots, n. \tag{6}$$

If the integral orbit of system (3) with the initial condition chosen from $\tau = 0$ to 1 is a periodic orbit justly and T chosen is the period of the system exactly, then Eq. (6) will be satisfied. At this time, $x^0$and T^0 are the solutions to be determined. However, this choice is almost impossible. In order to obtain the values of x and T that satisfied Eq. (6) at $\tau = 1$, they must be worked out by the iterative method. The details of the iteration arithmetic are as follows.

Integrating system (3) over one period and calculating the residual function r, in terms of the initial condition, yields

$$r_i(\eta, T^0) = x_i^1(\eta, T^0) - x_i^0 = x_i^1(\eta, T^0) - \eta_i, \quad i = 1, 2, \ldots, n. \tag{7}$$

Expand r into a Taylor series near η and T, and retain the linear teams only

$$r_i(\eta, T^0) + \sum_{s=1}^{n} \left[\frac{\partial r_i(\eta, T^0)}{\partial \eta_s} \Delta \eta_s \right] + \frac{\partial r_i(\eta, T^0)}{\partial T^0} \Delta T^0 = 0, \quad i = 1, 2, \ldots, n. \tag{8}$$

From Eq. (7), partial derivative of r_i with respect to η_j and T^0, the following formula can be obtained:

$$\frac{\partial r_i(\eta, T^0)}{\partial \eta_j} = \frac{\partial x_i^1(\eta, T^0)}{\partial \eta_j} - \frac{\partial \eta_i}{\partial \eta_j} = \frac{\partial x_i^1(\eta, T^0)}{\partial \eta_j} - \delta_{ij}, \quad i,j = 1, 2, \ldots, n, \tag{9}$$

where δ_{ij} is the Kronecker symbol, defined by

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{10}$$

$$\frac{\partial r_i(\eta, T^0)}{\partial T^0} = \frac{\partial x_i^1(\eta, T^0)}{\partial T^0}, \quad i = 1, 2, \ldots, n. \tag{11}$$

Taking partial derivative of Eq. (3) with respect to η_j, we have

$$\frac{\mathrm{d}}{\mathrm{d}\eta_j}\left(\frac{\mathrm{d}x_i}{\mathrm{d}\tau}\right) = T^0 \frac{\mathrm{d}}{\mathrm{d}\eta_j}[f_i(x, T\tau, \alpha)], \quad i,j = 1, 2, \ldots, n. \tag{12}$$

Exchanging the order of the partial differentiating, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}x_i}{\mathrm{d}\eta_j}\right) = T^0 \sum_{s=1}^{n} \left[\frac{\partial f_i(x, \alpha)}{\partial x_s} \frac{dx_s}{d\eta_j} \right], \quad i,j = 1, 2, \ldots, n. \tag{13}$$

Taking partial derivative of Eq. (3) with respect to T^0, we have

$$\frac{\mathrm{d}}{\mathrm{d}T^0}\left(\frac{\mathrm{d}x_i}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}}{\mathrm{d}T^0}\left[T^0 f_i(x, T^0\tau, \alpha)\right], \quad i,j = 1, 2, \ldots, n. \tag{14}$$

Exchanging the order of the partial differentiating, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}x_i}{\mathrm{d}T^0}\right) = f_i(x, T^0\tau, \alpha) + T^0\left[\sum_{j=1}^{n}\left(\frac{\partial f_i(x, T^0\tau, \alpha)}{\partial x_j}\frac{\mathrm{d}x_j}{\mathrm{d}T^0}\right)\right] + T^0\frac{f_i(x, T^0\tau, \alpha)}{\partial T^0}, \quad i = 1, 2, \ldots, n. \tag{15}$$

Let $\mathrm{d}x_i/\mathrm{d}\eta_j = y_{ij}$. Then Eq. (13) is transformed into

$$\frac{\mathrm{d}y_{ij}}{\mathrm{d}\tau} = T^0\sum_{s=1}^{n}\left[\frac{\partial f_i(x, T^0\tau, \alpha)}{\partial x_s}\frac{\mathrm{d}x_s}{\mathrm{d}\eta_j}\right] = T^0\left[\frac{\partial f_i(x, T^0\tau, \alpha)}{\partial x_1}y_{1j} + \frac{\partial f_i(x, T^0\tau, \alpha)}{\partial x_2}y_{2j} + \cdots + \frac{\partial f_i(x, T^0\tau, \alpha)}{\partial x_n}y_{nj}\right], \quad i,j = 1, 2, \ldots, n. \tag{16}$$

Let $\mathrm{d}x_i/\mathrm{d}T = y_{iT}$. Then Eq. (15) is transformed into

$$\frac{\mathrm{d}y_{iT}}{\mathrm{d}\tau} = f_i(x, T^0\tau, \alpha) + T^0\sum_{j=1}^{n}\left[\frac{\partial f_i(x, T^0\tau, \alpha)}{\partial x_j}y_{iT}\right] + T^0\frac{f_i(x, T^0\tau, \alpha)}{\partial T^0}, \quad i = 1, 2, \ldots, n. \tag{17}$$

Combining Eqs. (16) and (17), an initial-value problem of an ordinary differential equation is formed. The initial condition in these equations, by means of the selection of x_i^0, can be set as $y_{ij}^0 = \delta_{ij}$ (δ_{ij} is the Kronecker define symbol) and $y_{iT}^0 = 0$. Applying the initial condition mentioned above to integrate Eqs. (16) and (17) from $\tau = 0$ to 1, y_{ij}^1 and y_{iT}^1 can be obtained. Those are the values of $\mathrm{d}x_i/\mathrm{d}\eta_i$ and $\mathrm{d}x_i/\mathrm{d}T^0$ at $\tau = 1$, where $i,j = 1, 2, \ldots, n$. Of course, during the solving process the values of $\partial f_i(x, T^0\tau, \alpha)/\partial x_j$ at time τ will be used. So during the procedure of solving Eqs. (16) and (17), original system (3) must be integrated under initial condition (4) chosen at the same time. The values of x at time τ can be solved, and further the values of $\partial f_i(x, T^0\tau, \alpha)/\partial x_j$ can be obtained. Substitute the obtained solution of $\mathrm{d}x_i/\mathrm{d}\eta_i$ and $\mathrm{d}x_i/\mathrm{d}T$ at $\tau = 1$ into Eqs. (9) and (11) separately. Then the values of $\partial r_i(\eta, T^0)/\partial \eta_j$ and $\partial r_i(\eta, T^0)/\partial T^0$can be obtained, where $i,j = 1, 2, \ldots, n$. Substituting $r_i(\eta, T^0)$, $\partial r_i(\eta, T^0)/\partial \eta_j$ and $\partial r_i(\eta, T^0)/\partial T^0$ into Eq. (8), a set of the nth order linear equations with $n+1$ variables ($\Delta\eta_1$, $\Delta\eta_2$, …, $\Delta\eta_n$, ΔT^0) is formed. As the number of variables is more than the number of equations, there are an innumerable groups of solutions. In order to solve for the values of $\Delta\eta_1$, $\Delta\eta_2$, …, $\Delta\eta_n$, ΔT^0 from the equations mentioned above, one variable must be fixed. But which variable is appropriate for this? In terms of the difference between the $n+1$ initial conditions chosen and the values of the period at the end point obtained after one period integral, the variable that must be fixed need to be selected. In r, if the value of r_k is the least, then it means the initial value of x_k selected is the one closest to the actual periodic orbit of the system. Therefore, choose the minimum r_i solved to be fixed. Then the initial condition related to it will be kept constant at the next iterative process, i.e., the column corresponding to the minimum r_k in the coefficient matrix of the linear equation group (8) will be deleted. If ΔT^0 in r_i is the minimum, the column corresponding to T^0 cannot be deleted, because the period T of the periodic orbit of the system is certain. After solving the values of $\Delta\eta_1$, $\Delta\eta_2$, …, $\Delta\eta_n$, ΔT^0, let $x_i^0 = \eta_i + \Delta\eta_i$, $T_{next}^0 = T^0 + \Delta T$. Repeat such a procedure until the precision requested is satisfied. Thus the periodic orbit and its period of system (3) can be obtained, and then the periodic orbit and its period of system (1) can be obtained by a reverse transform $\tau = t/T$.

The procedure above can be summarized as follows:

1. Select the initial vector $x_i^0 = \eta_i$ $(i = 1, 2, \ldots, n)$ and an initial period T.
2. Integrate Eq. (3) from $\tau = 0$ to 1. The value at x_i^1 $(i = 1, 2, \ldots, n)$ can be obtained.
3. Calculate the residual vector r. Stop if the precision is satisfied, otherwise go to the next step.
4. Determine the minimum residual vector r except ΔT and mark it k.
5. Integrate Eqs. (16) and (17) from $\tau = 0$ to 1.
6. Solve Eq. (8). The values of $\Delta\eta_1$, $\Delta\eta_2$, …, $\Delta\eta_n$, ΔT will be obtained.
7. Let $x_i^0 = \eta_i + \Delta\eta_i$ $(i = 1, 2, \cdots, n)$ and $T^0 = T^0 + \Delta T$.
8. Go back to (2).

2.2. The stability of the periodic solution

Either the stable periodic solution or the unstable periodic solution of the non-linear system can be obtained by using the classical shooting method [9]. For system (3), if the integral direction is from $\tau = 0$ to 1, the periodic solution obtained is stable. If the integral direction is from $\tau = 0$ to -1, the periodic solution obtained is unstable. In order to determine the stability of the periodic solution obtained, in this paper Floquet stability analysis is used. Floquet stability analysis is used in determining linear stability of the periodic solution of a given non-linear system and it solves using the procedure given in Section 2.1. That is a very useful tool in determining the mode by which a period solution loses stability and what type of bifurcation may be anticipated.

Suppose a periodic solution has been already determined, that is, the values of η_1, η_2, …, η_n and T have been already obtained. Then for every circulation along the periodic track (limit circle) and the fixed period T, there exists

$$x^{next}(0) = \eta^{next} = \varphi(\eta_1, \eta_2, \ldots, \eta_n, T). \tag{18}$$

Apparently, formula (18) can be regarded as the iteration procedure of the variable η_1, η_2, …, η_n. Integrating Eq. (3) from $\tau = 0$ to 1 once corresponds to iterating one time. Hence, $(\eta_1, \eta_2, \ldots, \eta_n)$ ought to be the fixed point of the iteration procedure. If the iteration procedure constructed in the vicinity of the point $(\eta_1, \eta_2, \ldots, \eta_n)$ is convergent, then the periodic solution that led from it is asymptotically stable. The astringency of the iteration procedure, i.e., the stability of the periodic solution, is determined by the eigenvalue of the linearization mapping φ in the point $(\eta_1, \eta_2, \ldots, \eta_n)$, viz. determined by the eigenvalue λ of the matrix B. The matrix B is Jacobi matrix of system (3):

$$B = \left\{ \frac{\partial\varphi_i}{\partial\eta_j} \right\} = \{p_{i,j}(1)\}, \quad i,j = 1, 2, \ldots, n, \tag{19}$$

where $p_{i,j}(1)$ is the value of the element of Jacobi matrix B at $\tau = 1$ for the selected valuable $x = \eta$. These eigenvalues are entitled multipliers, and the numeral $\mu_i = \log\lambda_i$ is entitled character exponent [6,9]. According to Floquet's stability theory, for all λ, if only one λ_i satisfies $|\lambda_i| = 1$ and the others do $|\lambda_i| < 1$, then the periodic solution is stable. When certain λ_i crosses the unit circle, the stability of the limited circle will change. The matrix B can be obtained in the iteration procedure in Section 2.1, its eigenvalue can be solved by Jacobi rotary transform method. Thereby the stability of the periodic solution can be determined according to the eigenvalues solved.

2.3. The periodic solution of the forced van der Pol equation

In order to verify the validity of the method depicted above, such method was used to determine the periodic orbit and period of the forced van der Pol equation. The results obtained are compared with those in Ref. [7] and of Runge–Kutta method.

The forced van der Pol equation is as follows:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = b\cos\omega t. \tag{20}$$

Eq. (20) can be written as the following form:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\mu(x_1^2 - 1)x_2 - x_1 + b\cos\omega t.\end{aligned} \tag{21}$$

Let $\tau = t/T$; Eq. (21) can be transformed into

$$\begin{aligned}\dot{x}_1 &= Tx_2,\\ \dot{x}_2 &= T(-\mu(x_1^2 - 1)x_2 - x_1 + b\,cos\,(\omega T\tau)).\end{aligned} \tag{22}$$

When $b = 9$, $\omega = 3.1416$ and $\mu = 4.27-7.25$, the system has the P-5 periodic solution [7]. Choose the system parameter as $b = 9$, $\omega = 3.1416$ and $\mu = 5.25$, and select the initial iterative vector $x^0 = (x_1, x_2) = (0, 0)$ and $T^0 = 11.0$, follow the method mentioned above, the periodic orbit and its period of the system can be solved through seven times iterations. The period is $T =$ 9.999976613 and the periodic orbit is illustrated in Fig. 1. Under the condition of the parameters selected above, the theoretical period of the system is $T = 5 \times (2\pi/\omega) = 9.999976616$. The value of the period T that is predicted by the shooting method is nearly identical to that obtained by

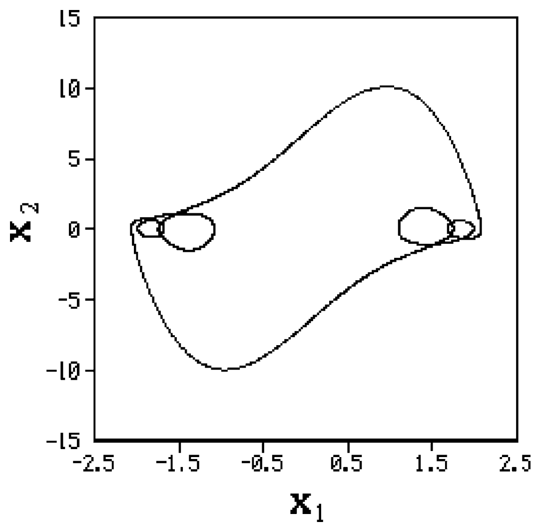

Fig. 1. Periodic orbit of the forced van der Pol equation determined through the method proposed in this paper. $b = 9$, $\omega = 3.1416$, $\mu = 5.25$.

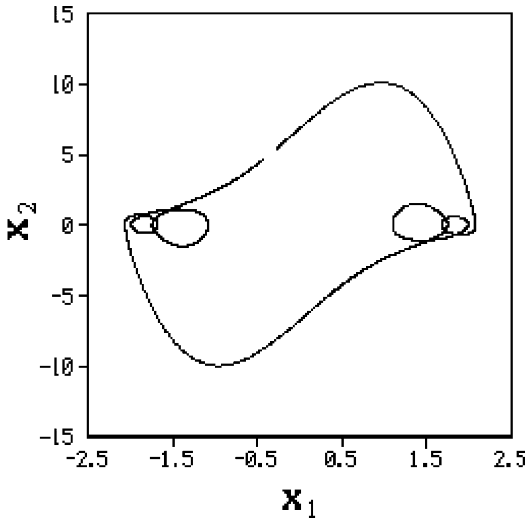

Fig. 2. Periodic orbit of the forced van der Pol equation determined through Runge–Kutta self-adaptive step integration. $b = 9$, $\omega = 3.1416$, $\mu = 5.25$.

theoretical analysis. A very small difference only in the ninth decimal place is found. Fig. 2 shows the periodic orbit determined by the self-adaptive step Runge–Kutta integration. By comparing the several results obtained above, obviously, the method depicted in this paper is valid and efficient.

3. The approximate analytic expressions of the periodic solution of the non-linear rotor–bearing system

3.1. The model of the non-linear rotor–bearing system

The mathematical model of a Jeffcott rotor–bearing system supported by the non-linear oil film [8] is shown in Fig. 3.

In Fig. 3, G denotes the half of the total weight of the rotor of the system, O denotes the geometry center of the axle bush, O_1 denotes the geometry center of rotor, O_c denotes the mass center of rotor. f_r and f_t denote the non-dimensional radial and tangential values of the non-linear oil film force, respectively. Selecting the non-dimensional eccentricity $e = \rho/c$ (ρ denotes the mass eccentricity of the rotor, c denotes the radius gap of the bearing), the equation of the system is

$$\ddot{x} = \frac{1}{\omega^2} - \frac{s}{\omega}(f_r \cos(\varphi) + f_t \sin(\varphi)) + e\cos(t),$$
$$\ddot{y} = -\frac{s}{\omega}(f_r \sin(\varphi) - f_t \cos(\varphi)) + e\sin(t), \tag{23}$$

where $s = 6\mu BR^3/(Mc^2\sqrt{gc})$, $\omega = \Omega\sqrt{c/g}$, $\varphi = \arctan(y/x)$, $t = \Omega\bar{t}$, μ is the lubricating oil viscosity, B the bearing width, R the bearing radius, M the bearing half mass, and Ω the bearing rps.

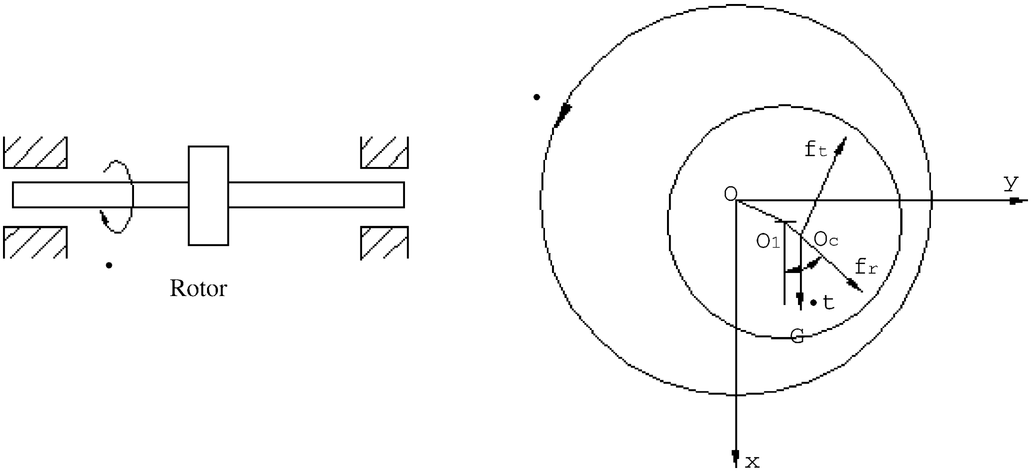

Fig. 3. Schematic of a Jeffcott rotor–bearing system.

Mark the state vector $(x_1, x_2, x_3, x_4) = (x, y, \dot{x}, \dot{y})$. Then Eq. (23) can be converted into the state equation

$$\begin{aligned}\frac{dx_1}{dt} &= x_3,\\ \frac{dx_2}{dt} &= x_4,\\ \frac{dx_3}{dt} &= \frac{1}{\omega^2} - \frac{s}{\omega}(f_r\cos(\varphi) + f_t\sin(\varphi)) + e\cos(t),\\ \frac{dx_4}{dt} &= -\frac{s}{\omega}(f_r\sin(\varphi) - f_t\cos(\varphi)) + e\sin(t),\end{aligned} \tag{24}$$

Adopt the infinite length bearing model, the expression of the radial and tangential values of the non-linear oil film forces of the bearing [8] are given by

$$\begin{aligned}f_r &= \left(1 - 2\frac{d\varphi}{dt}\right)\frac{2\varepsilon^2}{(2+\varepsilon^2)(1-\varepsilon^2)} + \frac{[\pi^2(2+\varepsilon^2) - 16]}{\pi(2+\varepsilon^2)(1-\varepsilon^2)^{3/2}}\frac{d\varepsilon}{dt},\\ f_t &= \left(1 - 2\frac{d\varphi}{dt}\right)\frac{\pi\varepsilon}{(2+\varepsilon^2)(1-\varepsilon^2)^{1/2}} + \frac{4\varepsilon}{(2+\varepsilon^2)(1-\varepsilon^2)}\frac{d\varepsilon}{dt}.\end{aligned} \tag{25}$$

3.2. The periodic orbit and period of the non-linear rotor–bearing system

As to Eq. (24), when the parameters are $s = 1.2$, $e = 0.2$ and $\omega = 0.8$, applying the method in Section 2.1, selecting the initial iterative vector $x^0 = (x_1, x_2, x_3, x_4) = (0.1, 0.2, 0.5, 0.5)$ and $T^0 = 5.6$, the periodic orbit and period of the system can be determined through 12 times iterations. The period $T = 6.2831903472$. The theoretical period of the system $T = 2\pi$ and the periodic orbit is shown in Fig. 4. And when the parameters are $s = 1.2$, $e = 0$ and $\omega = 1.2$, the system is a balanced rotor system and an autonomous system. We select the initial iterative vector $x^0 = (x_1, x_2, x_3, x_4) = (0.1, 0.2, 0.5, 0.5)$ and $T^0 = 5.6$, The periodic orbit and its period of the

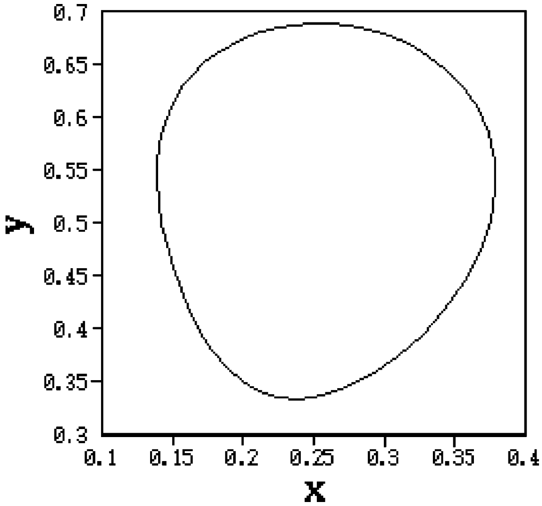

Fig. 4. Periodic orbit of the eccentric rotor–bearing system. $s = 1.2$, $e = 0.2$, $\omega = 0.8$.

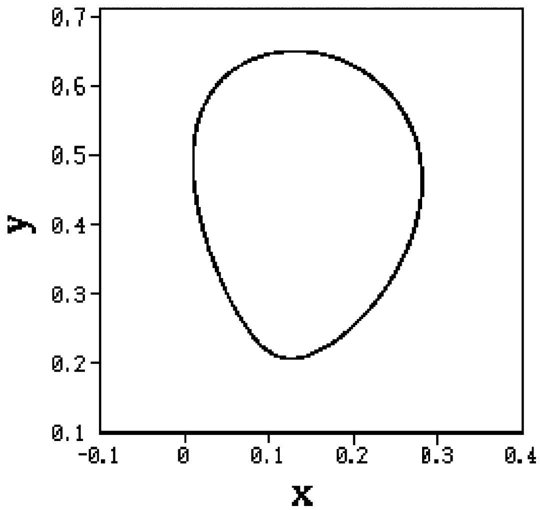

Fig. 5. Periodic orbit of the balance rotor–bearing system. $s = 1.2$, $e = 0$, $\omega = 1.2$.

system can be determined through nine times iterations. The period $T = 6.5219462376$, and the periodic orbit is shown in Fig. 5.

3.3. Determining the approximate analytic expressions of the periodic solution of the non-linear rotor–bearing system by using the harmonic balancing method

In order to calculate and deduce the following formulas conveniently, Eq. (23) is transformed in the linear method. Namely, the primary co-ordinate system of Eq. (23) will be rotated at θ angle.

Then Eq. (23) is transformed into

$$\ddot{x} = \frac{1}{m} f_x(x, y, \dot{x}, \dot{y}) + u \sin(t),$$
$$\ddot{y} = g + \frac{1}{m} f_y(x, y, \dot{x}, \dot{y}) + u \cos(t). \tag{26}$$

Eq. (26) is a strongly non-linear system. Its expressions of the oil film force are complicated. It is almost impossible to determine with precision the analytic expressions of the periodic solution of Eq. (26). Now, the approximate analytic expressions of the periodic solution of Eq. (26) with Galerkin forms are determined. Because Eq. (26) is a strongly non-linear system, a trigonometric function series as the eigenfunction of its periodic solution is taken. Suppose its n_s order approximate expressions of the stable periodic solution are given by

$$x(t) = a_{x0} + \sum_{n=1}^{n_s} (a_{xn} \cos(npt) - b_{xn} \sin(npt)),$$
$$y(t) = a_{y0} + \sum_{n=1}^{n_s} (a_{yn} \cos(npt) - b_{yn} \sin(npt)). \tag{27}$$

The character of the solution defines the parameter p. When $p = 1$, Eq. (27) determines the T-periodic solutions of the system. When $p \neq 1$, Eq. (27) determines the T/p-periodic solutions.

Then

$$\dot{x}(t) = \sum_{n=1}^{n_s} np(-a_{xn} \sin(npt) - b_{xn} \cos(npt)),$$
$$\dot{y}(t) = \sum_{n=1}^{n_s} np(-a_{yn} \sin(npt) - b_{yn} \cos(npt)), \tag{28}$$

$$\ddot{x}(t) = \sum_{n=1}^{n_s} n^2p^2(-a_{xn} \cos(npt) + b_{xn} \sin(npt)),$$
$$\ddot{y}(t) = \sum_{n=1}^{n_s} n^2p^2(-a_{yn} \cos(npt) + b_{yn} \sin(npt)). \tag{29}$$

Define $f_x = f_x(x, y, \dot{x}, \dot{y}), f_y = f_y(x, y, \dot{x}, \dot{y})$. Because the oil film forces are concerned with the track of the movement of the journal and the periodic track of Eq. (26) is defined by Eq. (27), the oil film forces f_x, f_y necessarily possess the same periodic characteristic as Eq. (27). So, the oil film forces f_x, f_y in Eq. (26) are expanded into the Fourier series with a form same as that of Eq. (27).

$$f_x = a_{fx0} + \sum_{n=1}^{n_s} (a_{fxn} \cos(npt) - b_{fxn} \sin(npt)),$$
$$f_y = a_{fy0} + \sum_{n=1}^{n_s} (a_{fyn} \cos(npt) - b_{fyn} \sin(npt)), \tag{30}$$

where

$$\begin{cases} a_{fx0} = \dfrac{p}{2\pi}\int_0^{2\pi} f_x \, \mathrm{d}\tau, \\ a_{fxn} = \dfrac{p}{\pi}\int_0^{2\pi} f_x \cos(np\tau) \, \mathrm{d}\tau, \quad n = 1, 2, \ldots, n_s, \\ b_{fxn} = \dfrac{p}{\pi}\int_0^{2\pi} f_x \sin(np\tau) \, \mathrm{d}\tau, \end{cases} \tag{31}$$

$$\begin{cases} a_{fy0} = \dfrac{p}{2\pi}\int_0^{2\pi} f_y \, \mathrm{d}\tau, \\ a_{fyn} = \dfrac{p}{\pi}\int_0^{2\pi} f_y \cos(np\tau) \, \mathrm{d}\tau, \quad n = 1, 2, \ldots, n_s. \\ b_{fyn} = \dfrac{p}{\pi}\int_0^{2\pi} f_y \sin(np\tau) \, \mathrm{d}\tau, \end{cases} \tag{32}$$

Substituting Eqs. (27)–(32) into Eq. (26) and equating the coefficients of the harmonic terms with the same order on both sides of the resultant equation yields the following relations between the unknown coefficients:

$$\begin{cases} \dfrac{p}{2\pi m}\int_0^{2\pi} f_x \, \mathrm{d}\tau = 0, \\ n^2p^2a_{xn} + \dfrac{p}{m\pi}\int_0^{2\pi} f_x \cos(np\tau) \, \mathrm{d}\tau = 0, \\ n^2p^2b_{xn} + \dfrac{p}{m\pi}\int_0^{2\pi} f_x \sin(np\tau) \, \mathrm{d}\tau - \delta u = 0, \quad n = 1, 2, \ldots, n_s. \\ \dfrac{p}{2\pi m}\int_0^{2\pi} f_y \, \mathrm{d}\tau + g = 0, \\ n^2p^2a_{yn} + \dfrac{p}{m\pi}\int_0^{2\pi} f_y \cos(np\tau) \, \mathrm{d}\tau + \delta u = 0, \\ n^2p^2b_{yn} + \dfrac{p}{m\pi}\int_0^{2\pi} f_y \sin(np\tau) \, \mathrm{d}\tau = 0, \end{cases} \tag{33}$$

Thereby, a non-linear algebraic equation set with $2(2n_s + 1)$ variables $(a_{x0}, a_{x1}, a_{x2}, \ldots, a_{xn_s}, b_{x0}, b_{x1}, b_{x2}, \ldots, b_{xn_s}, a_{y0}, a_{y1}, a_{y2}, \ldots, a_{yn_s}, b_{y0}, b_{y1}, b_{y2}, \ldots, b_{yn_s})$ can be formed. Solving the equation set above, approximate expressions of the periodic solution of Eq. (26) can be obtained.

So, the problem of determining the stable periodic solution of the differential Eq. (26) is transformed into that of solving the non-linear algebraic equation set Eq. (26).

The Newton–Raphson method can be used to solve Eq. (33). In order to overcome the shortcomings of the method that is crucial dependence on the initial iteration values, the continuation method to extend the convergence range is adopted in practice. The method does not confine the initial iteration values strictly. Select a set of the initial values. Then by solving the homotopy equation selected, the initial iteration values fine enough and that locate in the attractive region of the Newton iteration method can be obtained. Then the relative precision solution of Eq. (26) can be obtained by using the Newton–Raphson method.

In order to quicken the solving speed of the harmonic balancing method, firstly, a probable periodic solution is obtained by using the method in Section 2 and this determines its outline characteristic. Because the method in this section depends on that in Section 2, the stability of the periodic solution obtained can be tested by using the method about the stability in Section 2.

The primary work is to calculate the Jacobi matrix $J = \left[\partial F(x)/\partial x\right]$ in practice. The solution of the calculation indicates that the calculation speed can be advanced effectively by introducing the FFT method. It is feasible to the rough initial iteration values to introduce the continuation method. In theory, the periodic solution with any precision can be obtained if n_s selected is big enough. The increment of n_s results in the increment of the order of Eq. (33) necessarily. That causes the increment of the solving time.

The approximate analytic expressions of the periodic orbit of system (23) are solved by using the method above. In order to solve Eq. (33), the character of the solution of system (23) must be known. Otherwise the parameter p of Eq. (33) is confirmed only by tentative. For Eq. (23), when $e = 0.2$ and $s = 1.2$, $\omega = 0.8$, according to the conclusion of Section 3.2, it is known that the period of the periodic orbit is the same as that of the eccentricity excitation, so the system has the $P-1$ periodic solution under the condition of the parameters. According to the character of the solution, the parameter p in Eq. (33) is $p = 1$. The second and third order approximate analytic expressions of the solution obtained by using the method above are as follows:

The second order approximate analytic expression is

$$\begin{aligned} x(t) = {} & 0.24055915 + 0.03168490\sin(t) + 0.11036733\cos(t) \\ & + 0.00454233\sin(2t) + 0.01832749\cos(2t) \\ y(t) = {} & 0.53406359 + 0.16873016\sin(t) - 0.04590171\cos(t) \\ & + 0.02656739\sin(2t) + 0.01161501\cos(2t). \end{aligned}$$

The third order approximate analytic expression is

$$\begin{aligned} x(t) = {} & 0.24055915 + 0.03168490\sin(t) + 0.11036733\cos(t) \\ & + 0.00454233\sin(2t) + 0.01832749\cos(2t) \\ & + 0.00100444\sin(3t) + 0.00544075\cos(3t), \\ y(t) = {} & 0.53406359 + 0.16873016\sin(t) - 0.04590171\cos(t) \\ & + 0.02656739\sin(2t) + 0.01161501\cos(2t) \\ & + 0.00273012\sin(3t) + 0.00204074\cos(3t). \end{aligned}$$

The orbit of the system determined by the second and third order approximate analytic expressions above is shown in Fig. 6.

When there is no eccentricity excitation, namely $e = 0$, system (23) is an autonomous system. With the change of the whirl speed of the rotor, the Hopf bifurcation occurs in the system. When $e = 0$ and $s = 1.2$, $\omega = 1.2$, there is a limit cycle in the system. According to the character of the solution obtained in Section 3.2, system (23) has the $P-1$ periodic solution in the cases above, so $p = 1$. The approximate analytic expressions of the limit cycle are obtained by using the method above.

The second order approximate expressions of the cycle are shown as follows:

$$\begin{aligned} x(t) = {} & 0.13533054 + 0.04111482\sin(\mathrm{t}) - 0.12254715\cos(t) \\ & + 0.00274443\sin(2t) + 0.01544856\cos(2t), \\ y(t) = {} & 0.43799260 - 0.21750180\sin(t) - 0.06176640\cos(t) \\ & - 0.00693076\sin(2t) + 0.01136908\cos(2t). \end{aligned}$$

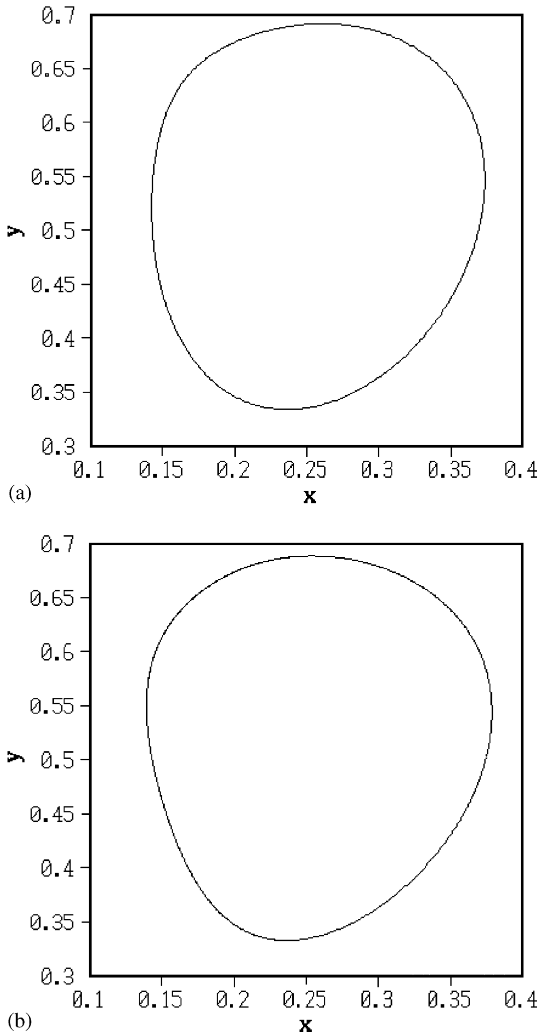

Fig. 6. The orbit of the non-autonomous system determined by the (a) second and (b) third order approximate analytic expressions. $s = 1.2$, $e = 0.2$, $\omega = 0.8$.

The third order approximate expressions of the cycle are shown as follows:

$$\begin{aligned} x(t) = &\, 0.13533054 + 0.04111482\sin(t) - 0.12254715\cos(t) \\ &+ 0.00274443\sin(2t) + 0.01544856\cos(2t) \\ &+ 0.00398165\sin(3t) - 0.00257291\cos(3t), \\ y(t) = &\, 0.43799260 - 0.21750180\sin(pt) - 0.06176640\cos(t) \\ &- 0.00693076\sin(2t) + 0.01136908\cos(2t) \\ &- 0.00884223\sin(3t) - 0.00002203\cos(3t). \end{aligned}$$

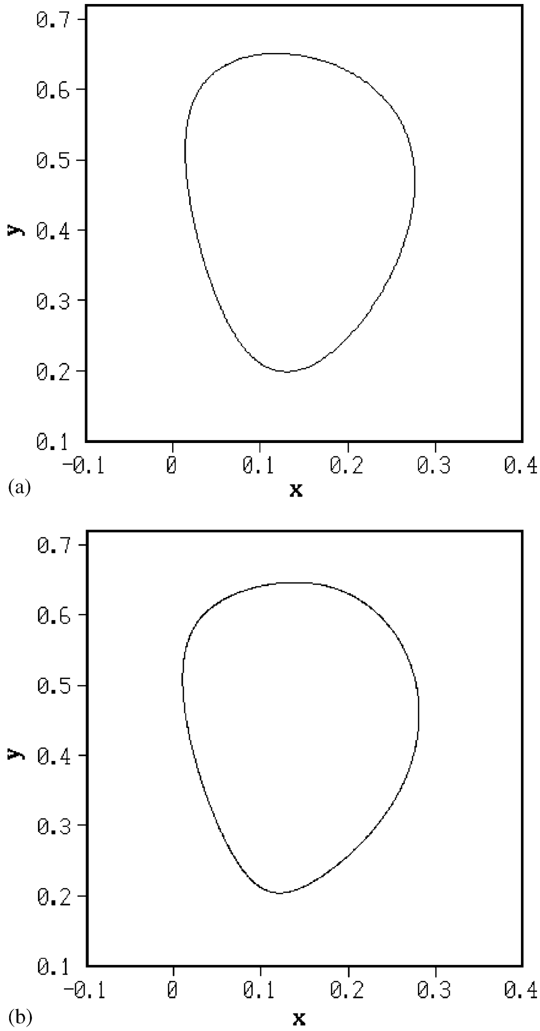

Fig. 7. The orbit of the autonomous system determined by the (a) second and (b) third order approximate analytic expressions. $s = 1.2$, $e = 0$, $\omega = 1.2$.

The orbit of the system determined by the second and third order approximate analytic expressions above is shown in Fig. 7.

4. Conclusions

The solving idea based on the traditional shooting method is adopted in this paper. The period T is used as one of the parameters to take part in the iteration in such method. The value of increment changed in iteration procedure is selected by using the optimization method. The procedure includes determining both of the periodic orbit and its period. So both of them can be obtained rapidly and accurately. The stability of the periodic solution is analyzed by using Floquet stability theory. The validity of the method is verified by determining the periodic orbit and its period of the forced van der Pol equation and a non-linear rotor–bearing system. The periodic orbit, period and the approximate analytic expression of the periodic solution of the non-linear rotor–bearing system are obtained. The method is efficiently used for the autonomous or non-autonomous system, and is also suitable for higher-dimensional system. The result of analysis can be used for the vibration control of the rotor–bearing system in practice.

Acknowledgements

This work was supported by the National Science Foundation of China, No. 19990510 (Grant), and the National Science Foundation of China, No. 10172067.

References

[1] M. Urabe, Galerkin's procedure for nonlinear periodic systems, *Archives for Rational Mechanics and Analysis* 20 (1965) 120–152.
[2] M. Urabe, A.A. Beiter, Numerical computation of nonlinear forced oscillations by Galerkin procedure, *Journal of Mathematical Analysis and Applications* 14 (1966) 107–140.
[3] S.L. Lau, Y.K. Chenung, S.Y. Wu, Incremental harmonic balance method with multiple time scales for a periodic vibration of nonlinear system, *American Society of Mechanical Engineers Journal of Mechanics* 50 (1983) 816–871.
[4] M.T. Xu, D.L. Cheng, A new approach to solving a type of vibration problem, *Journal of Sound and Vibration* 177 (4) (1994) 565–571.
[5] P. Sundararajan, S.T. Noah, Dynamics of forced nonlinear systems using shooting/arc-length continuation method application to rotor system, *American Society of Mechanical Engineers Journal of Vibration and Acoustics* 199 (1997) 9–20.
[6] J.Q. Zhou, Y.Y. Zhu, *Nonlinear Vibration*, The Press of Xi'an Jiaotong University, Xi'an, 1998.
[7] J.X. Xu, J. Jiang, The global bifurcation characteristics of the forced van der Pol oscillator, *Chaos Solitons and Fractals* 7 (1) (1996) 3–19.
[8] J.L. Gu, *Dynamics of Rotor*, The Press of National Defence Industry, Beijing, 1985.
[9] D.Y. Chai, F.S. Bai, *Altitude Numerical Analyses*, The Press of Tsinghua University, Beijing, 1997.